Relative Stability of Black Hole Threshold Solutions

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Outline

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• Spherically symmetric Einstein / SU(2) Yang-Mills (EYM)

• Critical phenomena (black hole threshold) review

• Relative stability of critical solutions

• Relative stability of scalar / YM Type II solutions

• (One) dynamical fate of $n=1$ Bartnik-McKinnon solution
Motivation

Why study Einstein-SU(2) Yang Mills?
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- Rich phenomenology in context of BH critical phenomena
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- Rich phenomenology in context of BH critical phenomena
- Provides good model in which to study relative stability of BH critical solutions
Spherically Symmetric SU(2) EYM
(with Eric Hirschmann)

- Consider SU(2) Yang-Mills (gauge) field, minimally coupled to Einstein gravity in spherical symmetry.

- General form for spherically symmetric metric \((G = c = 1)\)

\[
ds^2 = (-\alpha^2 + a^2 \beta^2) \ dt^2 + 2a^2 \beta \ dt \ dr + a^2 \ dr^2 + r^2 b^2 \ d\Omega^2
\]

where \(\alpha, \beta, a, b\) and \(R\) are functions of \(r\) and \(t\); \(R\) measures proper surface area ("areal radius")

- Gravitating mass well defined in spherically symmetry (at least in vacuum regions)

\[
m(R, t) = \frac{1}{2} R (1 - R^{;\mu} R_{;\mu})
\]

\(m, \ dm/dR\) are useful diagnostic quantities.
• Action for general Einstein / Yang-Mills theory

\[ S = \int d^4x \sqrt{-g} \left[ \frac{R}{16\pi G} - \frac{1}{g^2} F^a_{\mu\nu} F^{a\mu\nu} \right] \]

where \( a \) is the group index and \( g \) is the YM coupling constant that will be set to unity after this slide

• Einstein field equations

\[ \frac{1}{16\pi} G_{\mu\nu} = T_{\mu\nu} = \frac{1}{g^2} \left( 2 F^a_{\mu\lambda} F^{a\lambda\nu} - \frac{1}{2} g_{\mu\nu} F^a_{\alpha\beta} F^{a\alpha\beta} \right) \]

• Yang-Mills field equations:

\[ D_\mu F^{a\mu\nu} = 0 \]

where \( D_\mu \) is the gauge-covariant/spacetime covariant derivative
Now specialize to SU(2)—most general spherically-symmetric parameterization of the gauge connection is \( (Witten, \text{PRL 38, 121 (1977)}) \)

\[
A = u\tau^r dt + v\tau^r dr + (w\tau^\theta + \tilde{w}\tau^\phi)d\theta \\
+ \left( \cot \theta \tau^r + w\tau^\phi - \tilde{w}\tau^\theta \right) \sin \theta d\phi
\]

where \( u, v, w \) and \( \tilde{w} \) are all functions of \( r \) and \( t \) and the \( \tau^a \) are the spherical projection of the Pauli spin matrices and form an anti-Hermitian basis for SU(2), satisfying

\[
[\tau^a, \tau^b] = \epsilon^{abc} \tau^c \quad a, b, c \in \{r, \theta, \phi\}
\]

Field strength is then

\[
F = \tau^r (\dot{v} - u') dt \wedge dr \\
+ [(\dot{w} - uw)dt + (w' - v\tilde{w})dr] \wedge (\tau^\theta d\theta + \tau^\phi \sin \theta d\phi) \\
+ [(\dot{\tilde{w}} + uw)dt + (\tilde{w}' + v\tilde{w})dr] \wedge (\tau^\phi d\theta - \tau^\theta \sin \theta d\phi) \\
- (1 - w^2 - \tilde{w}^2)\tau^r d\theta \wedge \sin \theta d\phi
\]

where \( \cdot \equiv \partial/\partial t, ' \equiv \partial/\partial r \)
Convenient to write EOM in first-order-in-time form; to this end define auxiliary variables

\[\Pi = \frac{a}{\alpha} [\dot{w} - u\tilde{w} - \beta(w' - v\tilde{w})]\]
\[\Phi = w' - v\tilde{w}\]
\[P = \frac{a}{\alpha} [\dot{\tilde{w}} + uw - \beta(\tilde{w}' + vw)]\]
\[Q = \tilde{w}' + vw\]
\[Y = \frac{b^2r^2}{2\alpha a} (\dot{v} - u')\]
Then have the following EOM for the YM field:

\[
\begin{align*}
\dot{\Phi} &= \left(\frac{\alpha}{a} \Pi + \beta \Phi\right)' + uQ - v \left(\frac{\alpha}{a} P + \beta Q\right) - \bar{w} \frac{2\alpha a}{b^2 r^2} Y \\
\dot{Q} &= \left(\frac{\alpha}{a} P + \beta Q\right)' - u\Phi + v \left(\frac{\alpha}{a} \Pi + \beta \Phi\right) + w \frac{2\alpha a}{b^2 r^2} Y \\
\dot{\Pi} &= \left(\frac{\alpha}{a} \Phi + \beta \Pi\right)' + uP - v \left(\frac{\alpha}{a} Q + \beta P\right) + \frac{\alpha a}{b^2 r^2} w(1 - w^2 - \bar{w}^2) \\
\dot{P} &= \left(\frac{\alpha}{a} Q + \beta P\right)' - u\Pi + v \left(\frac{\alpha}{a} \Phi + \beta \Pi\right) + \frac{\alpha a}{b^2 r^2} \bar{w}(1 - w^2 - \bar{w}^2) \\
\dot{Y} &= \frac{\alpha}{a}(\bar{w}\Phi - wQ) + \beta(\bar{w}\Pi - wP) \\
Y' &= \bar{w}\Pi - wP \\
u' &= -\frac{2\alpha a}{r^2} Y
\end{align*}
\]
SU(2) EYM—Purely Magnetic Ansatz

- Assume electric charge density is identically 0; \( \implies Y(r, t) \equiv 0 \)

- Can set \( v = 0 \) by gauge transformation; \( Y = 0 \) then implies \( u = \text{const.} \). Further gauge transformation makes \( u = 0 \); EOM then imply that we can set \( \tilde{w} = 0 \) without loss of generality (i.e. that \( \tilde{w} \) is pure gauge in this case)

- Thus, in the context of the (dynamically self-consistent) “purely magnetic” ansatz, the dynamics of the YM field is described by the single “field”, \( w(r, t) \)

- Regularity at the origin, and finite-energy require that \( w(r, t) \) be in one of two vacuum states at \( r = 0 \) and \( r = \infty \):

\[
\begin{align*}
w(0, t) &= \pm 1 \\
\quad w(\infty, t) &= \pm 1
\end{align*}
\]

- Hereafter, will also work in polar/areal (Schwarzschild-like) coordinates

\[
ds^2 = -\alpha^2 dt^2 + a^2 dr^2 + r^2 d\Omega^2
\]
• Equations of motion simplify considerably:

\[
\begin{align*}
\dot{\Phi} &= \left(\frac{a}{\alpha} \Pi\right)', \\
\dot{\Pi} &= \left(\frac{a}{\alpha} \Phi\right)' + \frac{\alpha a}{r^2} w \left(1 - w^2\right), \\
\frac{a'}{a} &= \frac{1 - a^2}{2r} + \frac{1}{r} \left(\Phi^2 + \Pi^2 + \frac{a^2}{2r^2} \left(1 - w^2\right)^2\right), \\
\frac{\alpha'}{\alpha} &= \frac{a^2 - 1}{2r} + \frac{1}{r} \left(\Phi^2 + \Pi^2 - \frac{a^2}{2r^2} \left(1 - w^2\right)^2\right), \\
w' &= \Phi
\end{align*}
\]

• Initial conditions

\[
\begin{align*}
w(0, r) &= f(r), \\
\dot{w}(0, r) &= g(r)
\end{align*}
\]

where in practice typically choose \(g(r)\) so that data is time-symmetric \((g \equiv 0)\), or (almost) purely ingoing (imploding).
SU(2) EYM—General $t$-dependent Spherical Ansatz

- Now allow for both electric/magnetic charge densities

- Can still set $v(t, r) \equiv 0$ via gauge transformation, but now must apparently retain both $u(t, r)$ and $\tilde{w}(t, r)$ in addition to $w(t, r)$, although there is clearly gauge freedom left in $u$, $w$, $\tilde{w}$ (e.g. no evolution equation for $u$, and will see “gauge” effects in animations to come)

- Regularity (YM field must again be in vacuum state at origin)

\[ \lim_{r \to 0} \left( w(t, r)^2 + \tilde{w}(t, r)^2 \right) = 1 + O(r^2) \]

- Via gauge freedom can take

\[
\begin{align*}
  w(t, 0) &= 1 + O(r^2) \\
  \tilde{w}(t, 0) &= O(r^2) \\
  u(t, 0) &= O(r^2)
\end{align*}
\]
• Equations of motion

\[
\begin{align*}
\dot{w} &= \frac{\alpha}{a} \Pi + u \tilde{w} \\
\dot{\tilde{w}} &= \frac{\alpha}{a} P - u w \\
\dot{\Pi} &= \left( \frac{\alpha}{a} w' \right)' + u P + \frac{\alpha a}{r^2} w \left( 1 - w^2 - \tilde{w}^2 \right) \\
\dot{P} &= \left( \frac{\alpha}{a} \tilde{w}' \right)' - u \Pi + \frac{\alpha a}{r^2} \tilde{w} \left( 1 - w^2 - \tilde{w}^2 \right) \\
u' &= -\frac{2 \alpha a}{r^2} Y \\
Y' &= \tilde{w} \Pi - w P \\
\frac{\alpha'}{\alpha} &= \frac{a^2 - 1}{2r} + 4 \pi r a^2 S_r^{\prime r} = \cdots \\
\frac{a'}{a} &= \frac{1 - a^2}{2r} + 4 \pi r a^2 \rho = \cdots
\end{align*}
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• Note: In all of calculations described below, outgoing radiation conditions (Sommerfeld conditions), possibly corrected by relevant non-differentiated terms, work well
Review of Black Hole Critical Phenomena

- Consider parameterized families of solutions to Einstein equations, typically coupled to one or more matter fields (but vacuum case can also be considered); focus on collapse of matter/energy and black hole formation

- Family parameter, $p$, viewed as “control parameter” for initial data, and hence for subsequent dynamical evolution
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- Demand that family “interpolates” through the black hole threshold, i.e. that there exists a critical value, $p = p^\star$, such that

1. $p < p^\star$: No black hole forms
2. $p > p^\star$: Black hole forms

- Empirically (and for some models, analytically) scenarios 1. and 2. characterized by long-time, stable “end-states” of evolution, may be only such states
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- Solution in near-critical regime $p \sim p^*$ $\equiv$ black hole critical phenomena

- Use “competition” (loosely, kinetic energy vs potential energy) inherent in collapse models, and fine-tuning to dynamically evolve to unstable critical solution
Critical Phenomenology

- Critical solutions $\mathcal{Z}^*$, *do* exist (for all models considered thus far) and are locally unique (in solution space sense and up to certain symmetry transformations)—details of initial data, parameterization irrelevant
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• Critical solutions belong to two broad classes, that can conveniently be labelled by behaviour of black hole mass at threshold (which can be viewed as an order parameter)

1. **Type I**: Black hole formation turns on at *finite* mass
2. **Type II**: Black hole formation turns on at *infinitesimal* mass
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- Near-critical solutions characterized by scaling of dimensionful quantities (defines additional critical exponents)
Critical Phenomenology

- Although unstable, critical solutions tend to be minimally so, in the sense of having one unstable mode in the context of perturbation theory.

- Growth factor (Lyapunov exponent), \( \text{Re}\lambda_1 \), of unstable mode can be immediately related to exponents in scaling relations.
Type I Critical Solutions

• Smallest BH has finite mass

• Model will generally have one (or more) intrinsic length scales that will set the minimum mass

• Critical solution exhibits time translational invariance
  1. Continuous: static
  2. Discrete: periodic, defines “exponent”, $\omega$

• Scaling law for, e.g., “lifetime” of near-critical configuration during dynamical evolution

$$\tau \sim \sigma \ln |p - p^*| \quad \sigma = [\text{Re} \lambda_1]^{-1}$$
Type I Critical Solutions

- Examples (all spherically symmetric)
  - magnetic EYM ($n = 1$ Bartnik-McKinnon solution)
  - real scalar field (unstable oscillons, Brady et al)
  - complex scalar field (unstable mini-boson stars, Hawley, Lai)
  - perfect fluid (neutron star models on unstable branch, Noble)
Type II Critical Solutions

- No minimum BH mass, arbitrarily small BHs possible
- Critical solution exhibits scale invariance
  1. **Continuous**: continuous self-similarity (CSS)
  2. **Discrete**: discrete self-similarity (DSS), defines "echoing exponent", ∆
- Scaling law for, e.g., BH masses from super-critical evolutions:

\[
\ln M_{BH} \sim \gamma \ln |p - p^*| \quad \gamma = [\text{Re} \lambda_1]^{-1}
\]
Type II Critical Solutions

- Examples (spherically symmetric)
  - massless scalar field: $\Delta \approx 3.44, \gamma \approx 0.37$
  - magnetic EYM: $\Delta \approx 0.74, \gamma \approx 0.20$
  - non-linear sigma models (Choptuik et al, Husa et al)
  - perfect fluid (Evans & Coleman, Neilsen, Noble)

- Examples (axisymmetric)
  - vacuum gravitational waves (Abraham & Evans)
  - massless scalar field with angular momentum (Pretorius et al)
Critical Collapse in Purely Magnetic EYM
(Choptuik, Chmaj, Bizon, PRL 77, 424, (1996))

- See both Type I and Type II transitions, depending on initial data.

- Roughly, get Type II transition if, during collapse, configuration becomes sufficiently relativistic (kinetic-energy dominated), i.e. so that self-interaction “potential” term in effective Lagrangian

\[
\frac{(1 - w^2)^2}{r^2}
\]

becomes negligible in comparison to kinetic terms \(w'^2, \dot{w}^2\).

- Within context of this ansatz, Bartnik and McKinnon demonstrated numerically existence of countable infinity of regular, static solutions, \(w_n(r), n = 1, 2, \cdots\), to EYM equations, where \(n\) counts number of zero crossings of \(w(r)\).

- Solutions have been extensively studied, generalized since
Critical Collapse in Purely Magnetic EYM

- Key facts
  1. $w_n$ has $n$ unstable perturbative modes in magnetic ansatz
  2. $w_n$ has $2n$ unstable perturbative modes in general ansatz

- In particular, $n = 1$ solution can, and does, act as Type I critical solution for appropriate initial data families

- As mentioned above, Type II solution characterized by $\Delta \approx 0.74$, $\gamma = [\text{Re}\lambda_1]^{-1} \approx 0.20$
\( n = 1 \) Bartnik-McKinnon Solution
EYM Collapse Animations

- **ANIMATION** of Type I collapse \((w(r, t))\)
- **ANIMATION** of Type II collapse \(((1 - w)/r)\)
Type I EYM Collapse

\[ w(r,t) \]

\[ \ln(r) \]

\[ t=0, 19, 38 \]

\[ 56, 75, 94 \]

\[ 112, 131, 150 \]

\[ -2, 0, 2, 4 \]

\[ -1, -0.5, 0, 0.5 \]

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Type II EYM Collapse
Relative Stability of Critical Solutions

**QUESTION:** Given that critical solutions are *unstable*—i.e., in perturbation theory, always have at least one unstable mode—how does matter of one type behave in presence of critical solution of another type of matter?
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- Can at least partially address this issue by considering *relative stability* of critical solutions as (loosely) defined below

- Will proceed via (approximate) solution of full field equations

- Presumably could also do perturbation theory (perhaps using results from full PDEs as input), but some evidence that pert. theory will not be as effective in the relative stability case

- Consider two fields
  \[ \Psi_1(r, t), \quad \Psi_2(r, t) \]
  where we are investigating the stability of \( \Psi_2 \) w.r.t. critical soln of pure-\( \Psi_1 \) model, \( \Psi_1^* \)

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2. Tune \( p \) so that \( \Psi_1(0, r; p^* ) \implies \Psi_1^* \)
3. (Minimally) couple \( \Psi_2(t, r) \), with parameterized initial data \( \Psi_2(0, r; q) \) such that support of \( \Psi_2(t, r; q) \) during evolution overlaps support of \( \Psi_1(t, r; p) \). Generically, for pure \( \Psi_2 \) evolution \( \Psi_2(0, r; q^* ) \) will generate critical solution \( \Psi_2^* \)

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4. Fix initial data \( \Psi_2(0, r; q) \) (i.e. fix \( q \)), then retune \( \Psi_1(0, r; p) \), determining \( p_q^* \) such that \( [\Psi_1(0, r; p_q^*), \Psi_2(0, r; q)] \) generates a black hole threshold solution

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5. Study solution phenomenology as function of \( q \), including limits \( q \to 0 \), \( q \to p^* \)
Relative Stability of Scalar & YM Type II Solns.

- YM field $w$: Adopt dynamical purely magnetic ansatz described above, pure EYM model admits Type II solution, $w^*_{II}$, with

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- Massless scalar field, $\phi$: Has Type II solution with EYM model admits Type II solution with

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  $$\Delta_S \approx 3.44 \quad \gamma_S \approx 0.37$$

- Initial data

  $$w(0, r; p) = p e^{-(r-c)^2/s^2}$$
  $$\phi(0, r; q) = q e^{-(r-C)^2/S^2}$$

  with constants $c, s, C$ and $S$ chosen to ensure dynamical overlap of the supports of the two fields; $w(0, r)$ and $\phi(0, r)$ chosen to produce ingoing initial data.
Relative Stability of Scalar & YM Type II Solns.

- Setting $\phi \equiv 0$, tune $p$ to $p^*$ such that $w(0, r; p^*) \Rightarrow w_{II}^*$
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- Fix $q$, $\phi(r, 0; q)$, retune $p$, $w(r, 0, p)$ to determine $p_q^*$ that generates critical solution
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- Fix $q$, $\phi(r, 0; q)$, retune $p$, $w(r, 0, p)$ to determine $p_q^*$ that generates critical solution

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- Note that the YM critical solution has the larger Lyapunov exponent, so naively, one might expect it to be unstable in the presence of the Type II scalar solution
Relative Stability of Scalar & YM Type II Solns.

- **Animation** of critical solution for $q = 0$
  - $[dm/dr]_{YM}$ and $[dm/dr]_S \equiv 0$

- **Animation** of critical solution for $q = 1.0 \times 10^{-5}$
  - $[dm/dr]_{YM}$ and $10 \times [dm/dr]_S$

- **Animation** of critical solution for $q = 3.0 \times 10^{-5}$
  - $[dm/dr]_{YM}$ and $[dm/dr]_S$

- **Animation** of critical solution for $q = 5.0 \times 10^{-5}$
  - $[dm/dr]_{YM}$ and $[dm/dr]_S$

- **Animation** of critical solution for $q = 1.0 \times 10^{-4}$
  - $[dm/dr]_{YM}$ and $[dm/dr]_S$
Relative Stability of Scalar & YM Type II Solns.

$[dm/dr]_{YM}$ and $[dm/dr]_S$
Relative Stability of Scalar & YM Type II Solns.

$[dm/dr]_\text{YM}$ and $[dm/dr]_\text{S}$ (detail)
Concluding Remarks

• Calculations suggestive of an entire hierarchy of critical solutions when one considers the case of coupling to all conceivable forms of matter
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• Relaxation of symmetry assumptions quite likely to lead to additional phenomenology
Concluding Remarks

• Calculations suggestive of an entire *hierarchy* of critical solutions when one considers the case of coupling to all conceivable forms of matter

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